In the absence of the author, the paper by Professor Dunkel was read by title only. Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles:

(1) The object of this paper was to show how certain problems concerning the envelopes of circles may be easily solved without the calculus, and the relation of such envelopes to caustics. By aid of the caustic other facts may be obtained without the calculus such as the radius of curvature and an expression for the length of arc of the envelope.

(2) This paper gave an explanation of a curious multiplication method reputed to be used by Russian peasants. [Cf. this MONTHLY, *1918*, 139.]

(3) Sun-spot data are typical of a great mass of physical data, of which intensities are observed and plotted as ordinates against the time. The same methods of analysis apply to all. The mathematical theory of these methods is very simple.

This paper reviewed the work of Schwabe, Wolf and Wolfer, Newcomb, Shuster, Larmor and Yamaga, Lockyer, Clough, Dagobert and Turner very briefly. The main part was devoted to an exposition of the methods of Shuster and Turner.

The conclusion of the paper made reference to a new method of attack developed by the author in connection with a closely related problem.

(4) This report by a member of the National Committee summarized the work that has been done by the committee and outlined its plans for the future.

<div align="right">PAUL R. RIDER, Secretary-Treasurer.</div>

ON THE CONSTRUCTION AND MODELLING OF ALGEBRAIC SURFACES.

By ARNOLD EMCH, University of Illinois.

1. **Introduction.** In a paper on a simple method of curve tracing, which appeared in this MONTHLY (*1917*, 168–172), I have shown how the method of generating curves by projective pencils could be utilized very effectively in many cases of graphic representation in which the customary method of plotting by computation from the equation is unpracticable, or, to say the least, very tedious.

It is obvious to anyone who is thoroughly familiar with all phases of curve tracing that, whenever practicable, the projective, or more generally the geometrographic method, which is either purely constructive, or makes use of a minimum of arithmetical work, is preferable to the method of plotting from the equation. Moreover the graphic method reveals at least something of the curve as a geometric organism, while the plot of the equation merely shows the form of the curve without any inner geometric content.

The graphic method should, of course, not reject taking into account those geometric properties which are more readily and rigorously revealed by an analytical discussion of the equation.

The same remarks apply to the construction and representation of surfaces which are defined either geometrically, or by their equations. In fact, there is no doubt about the superiority of the graphic method when a model of the surface is required.

In analogy with the method used in my paper on curve tracing referred to above, I shall make use of the theorem that every algebraic surface may be generated by two projective pencils of certain surfaces of lower order. If the equation of the surface is given, we may put it in the form

(1) $P \cdot S - Q \cdot R = 0,$

in which P, Q, R, S are polynomials representing surfaces of lower order. The surface may now be generated by either of the two sets of projective pencils

$$P + \lambda Q = 0,$$
(2)
$$R + \lambda S = 0,$$

$$P + \lambda R = 0,$$
(3)
$$Q + \lambda S = 0.$$

For every value of the parameter λ the corresponding surfaces of the two projective pencils of a set intersect in a generatrix curve of the given surface.

The effectiveness of the projective method for the construction and modelling of surfaces, when their equations are known, lies in the proper choice of the polynomials P, Q, R, S. In the following examples the applicability of this method will be demonstrated.

2. Construction of a quintic surface with a given quartic nodal curve.
Let the quartic nodal curve D be defined as the intersection of the sphere

(1) $x^2 + y^2 + z^2 - r^2 = 0$

with the cylinder

(2) $x^2 + y^2 - rx = 0,$

so that D has an ordinary node at $U(r, 0, 0)$, Fig. 1. In this figure only the portion of D below the xy-plane is shown; it projects as a parabolic arc upon the xz-plane.

The equation

(3) $r(x^2 + y^2 + z^2 - r^2)^2 - z(x^2 + y^2 - rx)^2 = 0$

evidently represents a quintic surface S_5 with D as a nodal curve.

The problem is to construct S_5. With λ as a variable parameter the quintic may be generated by the two projective pencils

(4) $\sqrt{r}(x^2 + y^2 + z^2 - r^2) + \lambda(x^2 + y^2 - rx) = 0,$

(5) $(x^2 + y^2 - rx)z + \lambda \sqrt{r}(x^2 + y^2 + z^2 - r^2) = 0,$

of quadrics and cubics, respectively. In fact by eliminating λ between (4) and (5), we get (3). All quadrics (4) and cubics (5) pass through D. For every value of λ the quadric (4) and the cubic (5) intersect in a generatrix curve C^* of S_5 of the sixth order. But as D is common to the quadric and cubic, C^* degen-
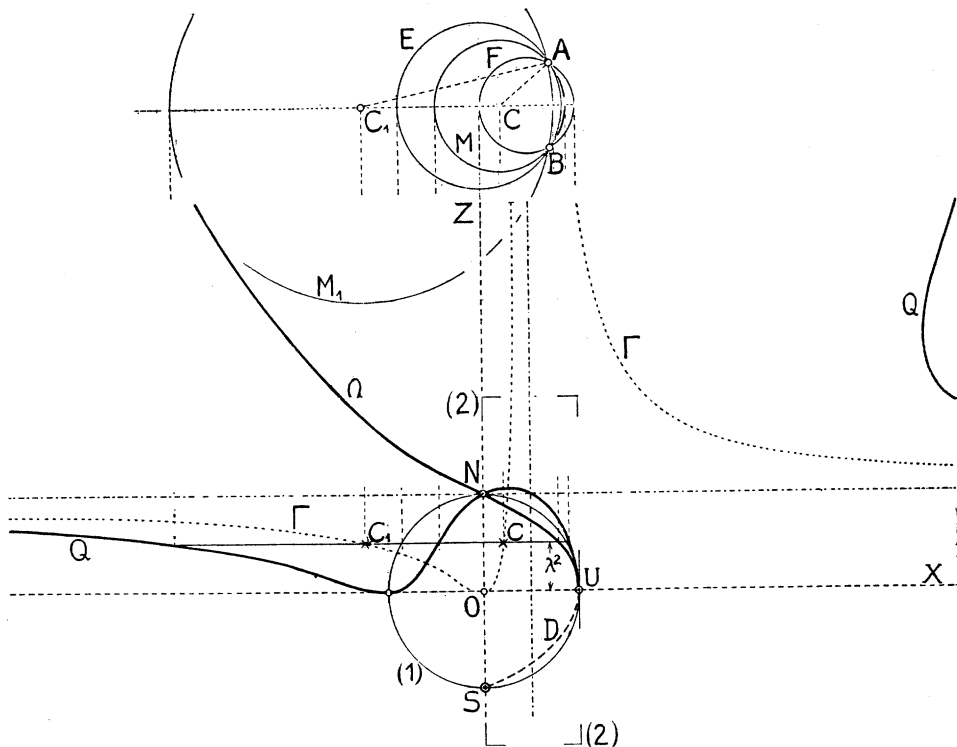

Fig. 1.

erates into D and a rest-curve M of order 2. To determine M, multiply (4) by λ and subtract from (5). After dividing the difference by the extraneous factor $x^2 + y^2 - rx$, the equation

(6) $$z - \lambda^2 = 0$$

remains which represents a plane parallel to the xy-plane. Thus M lies in this plane, and its equation is obtained by eliminating z between (6) and either (4) or (5). After some reductions this equation may be written in the form

(7) $$\left(x - \frac{\lambda r}{2(\sqrt{r} + \lambda)} \right)^2 + y^2 = \frac{4\sqrt{r}(\sqrt{r} + \lambda)(r^2 - \lambda^4) + \lambda^2 r^2}{4(\sqrt{r} + \lambda)^2}.$$

The rest curve M is therefore a circle in the plane $z = \lambda^2$. By letting λ vary, the circle M generates S_5. But there is another circle M_1 in the same plane, which is obtained by replacing in (7) λ by $-\lambda$, so that this circle M_1 is determined by

$$(8) \qquad \left(x + \frac{\lambda r}{2(\sqrt{r} - \lambda)}\right)^2 + y^2 = \frac{4\sqrt{r}(\sqrt{r} - \lambda)(r^2 - \lambda^4) + \lambda^2 r^2}{4(\sqrt{r} - \lambda)^2}.$$

The centers of these circles lie on a rational cubic Γ in the xz-plane with the parametric equations

$$(9) \qquad x = \frac{\pm \lambda r}{2(\sqrt{r} \pm \lambda)}, \qquad z = \lambda^2,$$

or, with the Cartesian equation

$$(10) \qquad z = \left(\frac{2\sqrt{r} \cdot x}{r - 2x}\right)^2.$$

The radii ρ of the circles $M(+\lambda)$ and $M_1(-\lambda)$ are determined by

$$(11) \qquad \rho = \frac{\sqrt{4\sqrt{r}(\sqrt{r} \pm \lambda)(r^2 - \lambda^4) + \lambda^2 r^2}}{2(\sqrt{r} \pm \lambda)}.$$

By means of formulas (9) or (10), and (11), and the planes $z = \lambda^2$, it is a simple matter to determine and locate these circles, and, consequently, to construct and model the surface. To exhibit the two sheets of the surface through D, the circles M and M_1 may be traced on equidistant glass-plates (as transparent as possible). In a plaster model some portion of the surface would be hidden from view. Computations may be considerably reduced by making a graph of Γ (10) in the xz-plane. The plane $z = \lambda^2$ intersects Γ in two points C and C_1, the centers of M and M_1, and the quartic D in two points A and B. CA and C_1A are the radii of M and M_1. A and B are the intersections of the circles E and F in which the plane $z = \lambda^2$ cuts the sphere (1) and the cylinder (2). By this simple construction, which is shown in horizontal projection in the upper portion of Fig. 1, the computation for the complicated expression for ρ may be avoided for all values $\lambda^2 \leqq r$.

From the construction as outlined above, we may merely conjecture as to the singularities of the surface. However, in order to complete the model properly, these must be determined definitely by discussing the equation of the surface with reference to its singularities. In the first place, there are evidently no real points of the surface below the xy-plane except those on the loops of D. The surface S_5 presents therefore the peculiarity that it contains as a part an isolated real branch of a curve which does not lie on the real film of the surface. The plane $z = 0$ cuts S_5 in the curve $(x^2 + y^2 - r^2)^2 = 0$ and is therefore a trope[1] of the surface; i.e., S_5 touches the xy-plane along the circle $x^2 + y^2 - r^2 = 0$. We may furthermore expect a singularity at the highest point $N(0, 0, r)$ of D. To determine its nature we transfer the origin to N, substitute $z' + r$ for z in (3), make the equation homogeneous, (replacing x, y, z' by x/t, y/t, z'/t, and multiplying through by t^5), and write the resulting equation in descending powers of t.

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[1] Basset, *Treatise on the geometry of surfaces*, London, 1910, pp. 23–25.

The factor multiplying the highest power of t represents the nodal cone at N, if it exists.   Treating (3) in this manner we get

$$t^3(4r^3z'^2 - r^3x^2) + \cdots = 0,$$

which shows that N is a binode with the biplanes

$$z - r = \pm \tfrac{1}{2}x.$$

Another singularity may be expected at the node $U(r, 0, 0)$ of the quartic D. Replacing x by $x' + r$ in (3) and proceeding as in case of the binode N, the equation becomes

$$t^3 \cdot 4r^2x'^2 + \cdots = 0,$$

which shows that U is an unode with $x = r$ as the uniplane.   In a similar manner we find that S is a binode with imaginary biplanes.[1]

Figure 1, Q, shows the quintic cross-section of the surface with the xz-plane, its equation being

(12) $$r(x^2 + z^2 - r^2)^2 - z(x^2 - rx)^2 = 0.$$

It has the line $z = r$ as an asymptote; N is an ordinary node, U a cusp.   The dotted curve is the locus Γ of the centers of circles M and M_1.   It has the lines $x = r/2$ and $z = r$ as asymptotes.

Another important question is whether it is possible that any of the circles M may degenerate into point circles.   This will be the case when the expression for ρ vanishes, *i.e.*, when

$$4\sqrt{r}(\sqrt{r} \pm \lambda)(r^2 - \lambda^4) + \lambda^2r^2 = 0.$$

Introducing $\lambda^2 = z$ this equation reduces to

(13) $$16z^5 - 16rz^4 - 24r^2z^3 + 31r^3z^2 + 8r^4z - 16r^5 = 0.$$

There are therefore 5 such circles.   But only one of these is real, which is determined by the root

(14) $$z = (1.0635\cdots)r.$$

The abscissa of this point circle is $x = (0.255\cdots)r$.   The point circle is at the intersection of Γ with Q.

3. **Construction of a cubic cyclide.**   Every cyclide may be generated by two projective pencils of spheres (of which one may degenerate into a pencil of planes). In case of a pencil of spheres and a projective pencil of planes the surface generated is a cubic cyclide.   Making use of the fact that a pencil of spheres is projective to the point set formed by the corresponding centers of these spheres, it is very easy to generate a cubic cylide which may be constructed and modelled without difficulty.

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[1] Basset, loc. cit., pp. 20–23.

Choose the line m $(y = 0, z = -c)$ as the locus of the centers of the spheres (Fig. 2). The pencil shall be determined by the condition that its spheres S shall pass through the circle $(z + c)^2 + y^2 = c^2$ in the yz-plane. Let the centers M of S be determined by the parameter $M_0M = \lambda$, so that the equation of S becomes

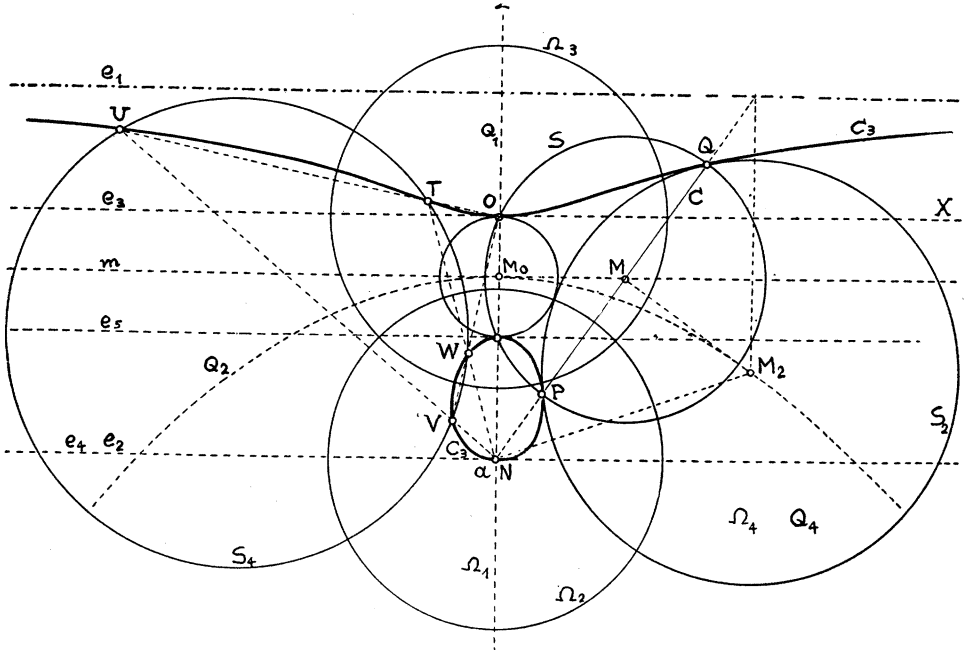(15) $$x^2 + y^2 + z^2 + 2cz - 2\lambda x = 0.$$

Fig. 2.

As the axis a of the projective pencil of planes choose the line $x = 0$, $z = -1/(2c)$, so that the pencil of planes through a and the centers of (15) becomes

(16) $$(2c^2 - 1)x + \lambda(2cz + 1) = 0.$$

For the same value of λ a plane (16) and a sphere (15) intersect in a circle C which generates the cyclide when λ varies. Eliminating λ between (15) and (16), we get for the equation of the cyclide

(17) $$2cz(x^2 + y^2 + z^2) + (4c^2 - 1)x^2 + y^2 + (4c^2 + 1)z^2 + 2cz = 0.$$

The construction of this cyclide is extremely simple. In Fig. 2 the circles C project as straight segments, like PQ, whose prolongation passes through N. The same figure also shows the cubic cross-section C_3 of the cyclide with the xz-plane. Fig. 3 is a picture of a plaster model of the cyclide.[1]

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[1] There seems to be no model of a cubic cyclide (which is not a cubic Dupin cyclide) in the market. The model represented by the figure was constructed by the author.

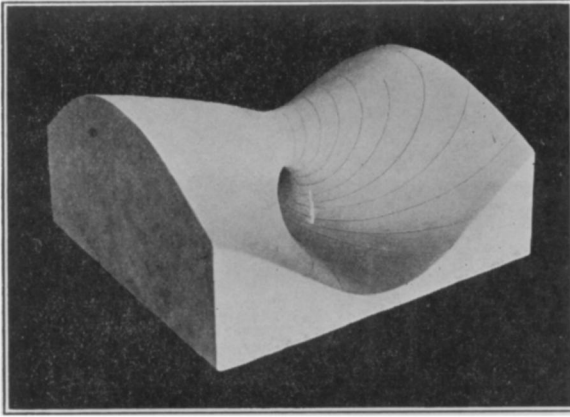The well-known properties of the cyclide may easily be shown on this surface. From equation (17) (making it homogeneous) follows that the infinite line i of the xy-plane lies on the cy-clide. There must therefore exist five pairs of lines on the cyclide which cut i, i.e., which are parallel to the xy-plane. In order to determine these, cut the cyclide by a plane $z = e$. As this plane passes through i, the rest curve of intersection with the cyclide is the conic

$$2ce(x^2 + y^2 + e^2) + (4c^2 -$$
$$1)x^2 + y^2 + (4c^2 + 1)e^2$$
$$+ 2ce = 0,$$

FIG. 3.

or

$$(2ce + 4c^2 - 1)x^2 + (2ce + 1)y^2 + 2ce^3 + (4c^2 + 1)e^2 + 2ce = 0.$$

This conic degenerates into a pair of lines when its discriminant vanishes, i.e.,

$$\begin{vmatrix} 2ce + 4c^2 - 1 & 0 & 0 \\ 0 & 2ce + 1 & 0 \\ 0 & 0 & 2ce^3 + (4c^2 + 1)e^2 + 2ce \end{vmatrix} = 0,$$

or when either of the three equations $2ce + 4c^2 - 1 = 0$, $2ce + 1 = 0$, $2ce^3 + (4c^2 + 1)e^2 + 2ce = 0$ is satisfied. This gives for e the five roots

$$e_1 = \frac{1 - 4c^2}{2c}, \qquad e_2 = -\frac{1}{2c}, \qquad e_3 = 0, \qquad e_4 = -\frac{1}{2c}, \qquad e_5 = -2c.$$

Each of the five planes $z = e_i$; $i = 1, 2, 3, 4, 5$ cuts the cyclide in a pair of lines. Those in $z = e_1$ are imaginary and parallel to the x-axis. In fact the line $y = 0$, $z = (1/2c) - 2c$, is an asymptote of C_3, hence $z = e_1$ is the tangent plane to the cyclide at the infinite point of this asymptote. The line $x = 0$, $z = -(1/2c)$, the axis a, must be counted as two pairs. The other two real pairs are easily obtained as

$$e_3 \begin{cases} z = 0, & \sqrt{1 - 4c^2} \cdot x + y = 0, \\ z = 0, & \sqrt{1 - 4c^2} \cdot x - y = 0, \end{cases}$$

$$e_5 \begin{cases} z = -2c, & x + \sqrt{1 - 4c^2} \cdot y = 0, \\ z = -2c, & x - \sqrt{1 - 4c^2} \cdot y = 0, \end{cases}$$

and obviously cross each other orthogonally.

The two pencils of planes through each of these pairs of lines cut the cyclide in a system of circles which belong to one of the five systems on the cyclide. These circles are, in general, cut out by double-tangent spheres which cut a fixed sphere (plane) Ω orthogonally, and whose centers lie on a quadric (conic) Q. The cyclide is *self-inverse* (for which Moutard[1] has proposed the bizarre word "anallagmatic," [2] which is now in general use) with respect to each of the spheres Ω. The five spheres Ω_i, $i = 1, 2, 3, 4, 5$, are mutually orthogonal and the quadrics Q are confocal. They may easily be determined by well-known methods.[3] For the sake of brevity I shall merely state the results:

$$z = e_1, \qquad \Omega_1 \equiv x = 0,$$

$$z = e_2, \qquad \Omega_2 \equiv x^2 + y^2 + \left(z + \frac{1}{2c}\right)^2 - \frac{1 - 4c^2}{4c^2} = 0,$$

$$z = e_3, \qquad \Omega_3 \equiv x^2 + y^2 + z^2 - 1 = 0,$$

$$z = e_4, \qquad \Omega_4 \equiv y = 0,$$

$$z = e_5, \qquad \Omega_5 \equiv x^2 + y^2 + (z + 2c)^2 + 1 = 0,$$

$$Q_1 \equiv x = 0 \text{ (plane)},$$

$$Q_2 \equiv cx^2 + 2(1 - 4c^2)(z + c) = 0 \text{ (parabola)},$$

$$Q_3 \equiv \frac{4c^2}{1 - 4c^2}x^2 - 4c^2y^2 + 4cz + 1 + 4c^2 = 0 \text{ (hyperbolic paraboloid)},$$

$$Q_4 \equiv y = 0 \text{ (plane)},$$

$$Q_5 \equiv 4c^2x^2 - \frac{4c^2}{1 - 4c^2}y^2 + 4cz + 1 = 0 \text{ (hyperbolic paraboloid)}.$$

For each of the five systems, the planes containing the circles of the system envelope quadric cones which have their vertices at the centers of the spheres Ω_i. In case of the cubic cyclide these cones degenerate into couples of pencils of planes through the corresponding pairs of lines on the cyclide. The spheres S_2, orthogonal to Ω_2 (and also to Ω_4), with their centers on the parabola Q_2, touch the cyclide along the circles C. The directrix of Q_2 is the line e_1, and its focus is N. The spheres S_4 orthogonal to Ω_4, with their centers on Q_4 (but not on Q_2) touch the cyclide in imaginary points. All spheres S_4 are also orthogonal to Ω_2. Among the ∞^2 S_4's there are ∞^1 spheres like the one shown in the figure, which are also orthogonal to Ω_3. Every sphere of this kind cuts the cubic C_3 in points T, U, V, W which, together with O and N, form a complete inscribed quadrilateral of the cubic. That there is a singly infinite number of such quadrilaterals through

[1] *Nouvelles Annales de Mathématiques*, vol. 22, 1864, pp. 306–309.

[2] (ά privatif, ἀλλασσω, je change.)

[3] Darboux, *Géométrie Analytique*, Paris, 1917, pp. 405–433, where a very clear and elementary account of the theory of cyclides in Cartesian coördinates is given.

O and N follows independently also from the fact that O and N form a *Steinerian couple*[1] on the cubic, *i.e.*, the points of tangency of two tangents from a point on the cubic (in this case the infinite point of C_3) to the same cubic. Like the cyclide itself, the cubic C_3 is anallagmatic with respect to O and N. In fact

$$OT \cdot OU = OW \cdot OV = 1; \qquad NV \cdot NU = NW \cdot NT = \frac{1 - 4c^2}{4c^2}.$$

In case of Ω_1 and Ω_4 the anallagmatic property reduces to symmetry with respect to these planes. In case of Ω_5 the anallagmatic constant (radius square of Ω_5) is $- 1$.

A CURVE OF PURSUIT.

By F. V. MORLEY, New College, Oxford University.

The curve of pursuit is one of that class of problems so entertainingly described by Professor David Eugene Smith, which in their travel through the centuries have preserved traces of the times of their proposers. The problem in one dimension, of the pursuer following the pursued in line, is common since the time of Zeno's paradox[2]; but the curve of pursuit does not seem to have been studied till the 18th century. An attempt has been made to make Leonardo da Vinci responsible, among his other wealth of contributions, for the statement of the problem.[3] But although it is quite possible to read into Leonardo's passage the essence of the question, it is perhaps doubtful that he ever had a conscious formulation. And of necessity, careful consideration of the problem had to wait until the methods of the calculus were known.

At any rate, the problem of the curve of pursuit was stated by Bouguer in 1732.[4] Although the days of the buccaneers were numbered, it is characteristic of the times that he chose for his example a privateer and a merchant vessel. Bouguer considered only the simplest case, where the pursued point moves along a line, but in the same volume of the *Mémoires de l'Académie Royale des Sciences* is a generalization of the problem by the remarkable de Maupertuis. Since then the problem in various guises has appeared in texts and periodicals.[5] One simple variant, in which the pursued point moves along a circle and the pursuer starts from the center, was re-proposed by Professor A. S. Hathaway in this MONTHLY, *1920*, 31. It is this case which is considered in this paper.

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[1] For the theory of Steinerian couples and quadruples on plane cubics see the author's *Introduction to Projective Geometry*, New York, 1905, pp. 197–204.

[2] See D. E. Smith, AMER. MATH. MONTHLY, Vol. 24, 1917, p. 64.

[3] Brocard, *Nouv. Corr. Math.*, Vol. 6, 1880, p. 211; cf. Loria, *Ebene Kurven*, 1902, p. 608.

[4] *Mémoires de l'Académie Royale des Sciences*, 1732.

[5] *E.g.*, *Math. Monthly* (ed. J. D. Runkle), 1, 1859, p. 249. There are also more elaborate papers such as "Sur les courbes de poursuite d'un cercle," by M. L. Dunoyer, *Nouv. Annales de Math.*, 4th series, Vol. 6, 1906, p. 193. [Compare page 91 of this issue.—EDITOR].